ON CERTAIN METHODS OF STURM AND THEIR APPLICATION TO THE ROOTS OF BESSEL'S FUNCTIONS.

BY PROFESSOR MAXIME BÔCHER.

Read at the February Meeting of the Society, 1897.

LAST November Mr. M. B. Porter, a graduate student at Harvard, submitted to me a proof that when $n > -\frac{1}{2}$ the well known theorem that between two successive positive roots of $J_n(x)$ lies at least one root of $J_{n+1}(x)$ can be extended to give the theorem that between two successive positive roots of $J_n(x)$ lies *just* one root of $J_{n+1}(x)$.* This proof consisted in applying to Bessel's equation the following proposition due to Sturm:†

If in a certain interval of the x-axis $\varphi_1(x) < \varphi_2(x)$ then between two successive roots, lying in this interval, of a solution of the equation:

$$\frac{d^2y}{dx^2} = \varphi_1(x) \cdot y,$$

there cannot lie more than one root of a solution of the equation:

$$\frac{d^2y}{dx^2} = \varphi_2(x) \cdot y.$$

The application to Bessel's functions is immediate when we let $y = \sqrt{x}\, J_n(x)$ for then y satisfies the differential equation:

$$\frac{d^2y}{dx^2} = \left(\frac{4n^2-1}{4x^2} - 1\right)y.$$

Not being aware at the time that the theorem above quoted is explicitly given by Sturm, Mr. Porter gave a proof of it which depends upon some better known theorems of the same mathematician. This proof, which is different from the one given by Sturm, is reproduced below (p. 210).

There has just appeared in the *American Journal*, (vol. xix, p. 75) another proof of the theorem concerning the

* For the sake of simplicity of statement we confine our attention to positive roots, the negative roots being numerically equal to them.

† *Liouville's Journal*, vol. i., p. 136. We will in future quote this article by merely mentioning Sturm's name.

roots of Bessel's functions by E. B. Van Vleck which also rests on a theorem of Sturm. This paper contains an extension of the theorems and methods in question to the subject of contiguous hypergeometric functions.

I propose to present now a third method for proving and extending the theorem, following out still other lines of thought marked out by Sturm (p. 160 seq.), and I will then compare the three methods showing wherein lie the peculiar advantages and possibilities of extension of each. My purpose has been to call attention to Sturm's methods rather than to elaborate the details of the theory of the roots of Bessel's functions. The very fact that the properties of the roots of Bessel's functions here considered are not generally known brings out in a striking way how little this fundamental paper of Sturm has really been read.

The relative position of the positive roots of $J_n(x)$ and $J_{n+1}(x)$ is evidently the same as that of the positive roots of $R_n(\xi)$ and $R_{n+1}(\xi)$ where $\xi = x^2/4$ and $R_n(\xi) = (2/x)^n J_n(x)$. The object of introducing $R_n(\xi)$ is that (cf. Gray and Mathews: *Treatise on Bessel Functions*, p. 46)

$$R_n'(\xi) = -R_{n+1}(\xi),$$

(the accent denoting differentiation) so that Rolle's theorem shows us that between two successive roots of $R_n(\xi)$ lies at least one root of $R_{n+1}(\xi)$. To show that there lies in this interval only one root of $R_{n+1}(\xi)$ we have merely to pass from Bessel's equation to the equation satisfied by $R_n(\xi)$:

$$\xi \frac{d^2y}{d\xi^2} + (n+1)\frac{dy}{d\xi} + y = 0.$$

A point at which $R_n'(\xi) = 0$ must be a maximum or a minimum of $R_n(\xi)$ as otherwise $R_n''(\xi)$ would also vanish at this point and this is seen from the differential equation to be impossible. If then between two successive vanishing points of $R_n(\xi)$ $R_n'(\xi)$ vanished more than once it would have to vanish at least three times and at one of these points $R_n(\xi)$ would have a minimum if positive, a maximum if negative. In either case $R_n(\xi)$ and $R_n''(\xi)$ would have the same sign at a point where $R_n'(\xi) = 0$, and this, as we see from the differential equation, is impossible. This completes the proof of the theorem that *between two successive positive roots of $J_n(x)$ lies one and only one root of $J_{n+1}(x)$.**

* Cf. Sturm, p. 162.

Precisely the methods just used enable us to obtain an analogous theorem, which the writer has never seen stated, concerning the roots of $J_{n+2}(x)$. We again consider two successive roots a and b $(0 < a < b)$ of $R_n(\xi)$. Between a and b lies, as we have seen, one and only one point c for which $R_n'(c) = 0$. From a to c, R_n and R_n' have the same sign so that we see from the differential equation that $R_n''(\xi)$ cannot vanish between a and c. $R_n''(\xi)$ must, however, vanish at least once in $a\ b$ (and, therefore, as we have just seen in $c\ b$) for $R_n(a) = R_n(b) = 0$, while $R_n'a$ and $R_n'(b)$ have opposite signs, so that it is clear from the differential equation that $R_n''(a)$ and $R_n''(b)$ have opposite signs. It remains to see whether R_n'' can vanish more than once in $c\ b$. It is easily seen that at every point where $R_n'' = 0$, R_n' has a maximum or a minimum so that if R_n'' vanishes more than once in $a\ b$ it vanishes at least three times, and, therefore, if R_n' is positive it has at least one minimum, if negative, one maximum. In either case R_n' and R_n''' have the same sign while $R_n'' = 0$, and this is seen to be impossible when we consider the following relation obtained by differentiating the differential equation for R_n:

$$\xi R_n'''(\xi) + (n+2) R_n''(\xi) + R_n'(\xi) = 0.$$

We thus get the theorem:

Between two successive positive roots of $J_n(x)$ lies one and only one root of $J_{n+2}(x)$.

This root is greater than the root of $J_{n+1}(x)$ which lies in the interval in question.

It is clear that the method here used may be applied to a large class of similar cases.* For example the following theorem may be deduced directly from Bessel's equation:

Between two successive roots of $J_n(x)$, between which $x > |n|$, lies one and only one root of $J_n'(x)$ and one and only one root of $J_n''(x)$. The root of $J_n''(x)$ is greater than the root of $J_n'(x)$.

* The principal theorem used applies even to the case of certain partial differential equations. It may be stated as follows (cf. for $n = 1$ Sturm p. 160).

In the partial differential equation:

$$\sum_{i=1}^{i=n} \left(a_i \frac{\partial^2 u}{\partial x_i^2} + b_i \frac{\partial u}{\partial x_i} \right) + cu = 0.$$

a_i, b_i, c are real functions of the real variables $x_1, \ldots x_n$, and throughout a certain region, $a_1, \ldots a_n$ are positive. If throughout this region c is positive u cannot have a positive minimum or a negative maximum; if c is negative u cannot have a positive maximum or a negative minimum.

For an example of the use of this theorem when $n = 2$ cf. Picard Traité d' Analyse, Vol. ii., p. 34.

We will now proceed to a comparison of the method just explained with the one used by Van Vleck. It seems to me that the former is of a distinctly more elementary character, inasmuch as it involves the sort of discussion which we must always go through with when we wish to get an idea of the nature of a real function of a real variable, viz., the location of maxima and minima, points of inflection, etc. Van Vleck's method, on the other hand, is not only extremely elegant but is in touch with the important idea of *related** differential equations so that it can be applied at once to hypergeometric functions and in fact to the solutions of any regular linear differential equation of the second order and possibly, therefore, ultimately to irregular equations of the second order in general, since these may be regarded as the limiting forms of regular equations. The method contained in the foregoing pages, while applying as readily to irregular as to regular equations, depends essentially on the properties of derivatives which are merely special cases of related functions (having in general accessory points†), and appears, therefore, at first sight to have a more limited range of application than Van Vleck's. The two methods appear, however, on closer examination, to be coextensive, at least in their application to Bessel's functions and hypergeometric functions. Two very simple examples may suffice here.

1. The theorem above proved that between two successive positive roots of $J_n(x)$ lies just one root of $J_{n+2}(x)$ follows at once by Van Vleck's method from the relation:

$$J_{n+2}J_{-n} - J_{-(n+2)}J_n = \frac{4(n+1)\sin(n+1)\pi}{\pi x^2},$$

or when n is an integer:

$$J_{n+2}Y_n - Y_{n+2}J_n = \frac{2(n+1)}{x^2}.$$

2. If we remember that

$$\frac{d}{dx}F(\alpha, \beta, \gamma, x) = \frac{\alpha\beta}{\gamma}F(\alpha+1, \beta+1, \gamma+1, x)$$

and that $F(\alpha, \beta, \gamma, x)$ satisfies the differential equation:

* In my lectures I have been in the habit of translating the German term *verwandt* by *kindred* rather than *related* as this last term is constantly used in an entirely untechnical sense to denote any kind of connection or similarity in form.

† Thus $J_n'(x)$ has accessory points at $x = \pm n$.

$$\frac{d^2y}{dx^2}+\frac{\gamma-(\alpha+\beta+1)x}{x(1-x)}\frac{dy}{dx}-\frac{\alpha\beta}{x(1-x)}y=0,$$

we obtain by the method above explained the theorem:

Between two successive roots of $F(\alpha, \beta, \gamma, x)$ which lie in the interval from 0 to 1 lies one and only one root of $F(\alpha+1, \beta+1, \gamma+1, x)$; and, provided that the point $x=\gamma/(\alpha+\beta+1)$ does not lie between the roots of $F(\alpha, \beta, \gamma, x)$ in question, one and only one root of $F(\alpha+2, \beta+2, \gamma+2, x)$.

The second part of this theorem might easily be obtained by Van Vleck's method, while the first part is merely a special case of his theorem concerning contiguous hypergeometric functions. The general theorem can easily be obtained in the same way.

It is clear that what is essential to the application of our method is that the two related functions should be so prepared by multiplication by suitable factors and, if necessary, by a change of the independent variable that one is a derivative of the other.

Very different from the two methods so far discussed and in some respects much more far-reaching is the method suggested by Mr. Porter. In order to simplify matters we will restrict ourselves in this article to positive values of n. Before beginning with a discussion of the method it will be well to note that the proof given at the beginning of this paper established the fact that, *k being any positive quantity, between two successive positive roots of $J_n(x)$ (n positive) cannot lie more than one root of $J_{n+k}(x)$.*

The theorem of Sturm quoted at the beginning of this paper was proved by Mr. Porter by means of the following two more fundamental theorems also due to Sturm:

I. *p and q being continuous real functions of the real variable x, between two successive roots of a solution of the differential equation:*

$$\frac{d^2y}{dx^2}+p\frac{dy}{dx}+qy=0$$

will lie one and only one root of any linearly independent solution.

II. *If, φ_1 and φ_2 being continuous real functions of the real variable x, $\varphi_1<\varphi_2$, and if y_1 and y_2 both vanish at a point x_0 and satisfy respectively the equations:*

$$\frac{d^2y}{dx^2}=\varphi_1\cdot y, \tag{1}$$

$$\frac{d^2y}{dx^2}=\varphi_2\cdot y, \tag{2}$$

*then, if y_2 has n roots to the right (left) of x_0, y_1, will also have at least n roots there and the k^{th} root of y_2 from x_0 will be greater (less) than the k^{th} root of y_1 from x_0**.

Mr. Porter's proof now is as follows:

Retaining the notation of the last theorem, let us suppose that between two successive roots of a solution y_1 of (1) lie more than one root of a solution $\bar{y}$ of (2). Let y_2 be a solution of (2) which vanishes at one of the two roots of y_1 in question. Then by Theorem I y_2 must vanish between any two successive roots of $\bar{y}$ and therefore at least once between the roots of y_1 in question; but this is impossible by Theorem II.

I have reproduced this proof not merely because I consider the method used instructive, but because it is capable of extension to cases which could not readily be attacked by the method given by Sturm.

Before going farther, I should like to present a proof of Theorem I just stated which I have been in the habit of giving in my lectures and to which, though it will be familiar to many, I am unable to give a reference†.

Let a and b be two successive roots of a solution y_1 of the differential equation. Let y_2 be any solution linearly independent of y_1 and consider the ratio y_1/y_2. If y_2 did not vanish between a and b this ratio would be continuous throughout the interval and, since it vanishes at the extremities, its derivative $(y_1' y_2 - y_1 y_2')/y_2^2$ would vanish in the interval. This is impossible as the vanishing of $y_1' y_2 - y_1 y_2'$ is obviously the condition that y_1 and y_2 should be linearly dependent. y_2 must then vanish at least once between two successive points where y_1 vanishes, and we see that it cannot vanish more than once, for if it did, by a similar proof y_1 would vanish between two successive points where y_2 vanishes and therefore a and b would not be *successive* roots of y_1.‡

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* Sturm, p. 125.

† Two methods of proof are given by Sturm, of which the second, depending, as it does, on the formula $y_1 y_2' - y_2 y_1' = ce^{-\int p dx}$ has a certain analogy to Van Vleck's method above referred to.

‡ Various modifications of this proof can be given, one of which will be suggested by a method used by Van Vleck on p. 76 of the paper above quoted. If p and q are analytic functions y_1/y_2 is the "Schwarzian s-function" and the following form of the proof is instructive:

If x moves along its axis of reals s will do the same (p, q, y_1, y_2 being supposed real for the real values of x in question). It is a fundamental property of the s-function that the representation of the x-plane on the s-plane is conformal except at the singular points of the differential equation. As x moves always in one direction from one real root of y_1 to the

It is clear that the proof just given really establishes the following generalized form of the theorem:

If within the interval ab (excluding the ends) the coefficients of the differential equation:

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$$

are continuous, and if there exists a solution y_1 which does not vanish between a and b and such that its ratio to a linearly independent solution y_2 approaches zero as we approach each end of the interval, then y_2 vanishes once and only once between a and b.

In particular if p and q are analytic functions a and b might be regular points with real exponents, and y_1 a solution corresponding to the largest of these exponents at each point. Here either a or b, or both, may, of course, be not merely regular but non-singular points.

Although the theorem just stated is capable of important general application, it will be sufficient for our purposes to deduce from it the following theorem concerning Bessel's functions:

If n and k are positive and a is the smallest positive root of $J_n(x)$ then $J_{n+k}(x)$ cannot vanish in the interval from o to a.

For if $J_{n+k}(x)$ vanishes at the point b between o and a the solution of Bessel's equation with parameter $n+k$ which vanishes when $x=a$ would vanish again by the theorem just proved between o and b (say at c) and therefore by the theorem numbered II above, $J_n(x)$ would vanish between c and a, *i. e.*, a would not be the smallest positive root of $J_n(x)$. It will be noticed that we have merely repeated Mr. Porter's proof under slightly different circumstances.

Since, then, as n increases the smallest positive root $J_n(x)$ increases* and, since between two successive positive roots of $J_n(x)$, lies at most one root of $J_{n+k}(x)$ (k positive) it fol-

next without passing a singular point s must move always in the same direction (else the representation would not be corformal) from $s=0$ back to $s=0$ and must, therefore, pass through the point $s=\infty$ once and only once. This establishes the theorem. This proof is implicitly contained in Klein's paper: "Ueber die Nullstellen der hypergeometrishen Reihe." *Math. Ann.*, vol. 37.

* We have, strictly speaking, only proved that this root does not decrease as n increases. It is, however, easy to see that it cannot remain constant, for $J_n(x)$ is an analytic function of n and x and, therefore, if $J_n(x)=0$ x is an analytic function of n and if it were constant for any continuous set of values of n it would be constant for all values of n. This, however, is not the case, as the smallest root of $J_n(x)$ is evidently greater than n, since $J_n(x)$ cannot have a positive maximum or a negative minimum when $x<n$.

lows at once that *as n (which is supposed positive) increases, all the positive roots of $J_n(x)$ increase.**

From this it appears that, when k is sufficiently small, between two successive positive roots of $J_n(x)$ lies one and only one root of $J_{n+k}(x)$, and the question arises how large k can be allowed to become without this condition of affairs changing. It is clear that the theorem can cease to be true only when one of the roots of $J_{n+k}(x)$ passes a root of $J_n(x)$, and this can occur only when each of the subsequent roots of $J_{n+k}(x)$ has passed one of the roots of $J_n(x)$ as otherwise we should have two roots of $J_{n+k}(x)$ between two successive roots of $J_n(x)$. It is, therefore, the large roots of $J_{n+k}(x)$ which will first reach the next roots of $J_n(x)$ and from the fact that the large roots are given approximately by the formula:

$$\frac{\pi}{4}(2n-1+4p),$$

where p takes on in succession all large integral values, it is clear that as long as $k<2$ no root of $J_{n+k}(x)$ will have reached the next root of $J_n(x)$, while when $k>2$ the large roots of $J_{n+k}(x)$ will have passed those of $J_n(x)$. What happens when $k=2$ can be determined by the use of more exact asymptotic values for the large roots. We will not, however, stop to consider this point, as we have already obtained the result we need by another method (p. 207). We have thus obtained the theorem:

If n is positive and $o<k\leqq 2$ between two successive positive roots of $J_n(x)$ lies one and only one root of $J_{n+k}(x)$ and vice versa.

This theorem contains (when $k=1$) the theorem established by Van Vleck as a special case. It brings out, however, clearly, at least when n is positive, (and it is herein that I see the chief importance of the method suggested by Mr. Porter) that *the theorem in question has no necessary connection whatever with the subject of related functions.*

The method just explained gives us, when carried a step farther, the following theorem:

If n is positive and $2p<k\leqq 2p+2$, where p is any positive integer, in each of the intervals bounded by successive positive roots of $J_n(x)$ lies one and only one root of $J_{n+k}(x)$ except in p of the intervals in which no root of $J_{n+k}(x)$ lies. Exceptions occur when and only when roots of $J_n(x)$ and $J_{n+k}(x)$ coincide, in which case one of the p intervals above mentioned is replaced by the two intervals which are separated by the common root in question.

* This theorem may also be deduced by following more closely the methods of Sturm. Cf. the remark of Schläfli in the foot-note on p. 137 of vol. 10, of the *Math. Ann.*

The precise position of these p intervals can be determined when k is an integer either by Van Vleck's method or by the method explained at the beginning of this paper. If, for instance, $k = 3$ we may proceed as follows. We easily find that R_n satisfies the relation:

$$\xi^2 R_n''' + [\xi - (n+1)(n+2)] R_n' - (n+2) R_n = 0.$$

At two successive points where $R_n = 0$ R_n''' will therefore have opposite signs unless between the points is question $\xi = (n+1)(n+2)$; and we have the theorem:

$J_{n+3}(x)$ vanishes once and only once between two successive positive roots of $J_n(x)$ except between the two roots which include between them the point $x = 2\sqrt{(n+1)(n+2)}$ in which interval $J_{n+3}(x)$ does not vanish at all.

Bessel's equation is clearly only a first example to which the methods of Sturm, which we have discussed, can be profitably applied. Further considerations of this sort, however, with reference especially to Bessel's functions with negative subscripts and to the theory of hypergeometric functions I will reserve for a future occasion. I shall be satisfied if the foregoing discussion helps to emphasize the importance of Sturm's paper.

HARVARD UNIVERSITY,
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ON THE TRANSITIVE SUBSTITUTION GROUPS WHOSE ORDERS ARE THE PRODUCTS OF THREE PRIME NUMBERS.

BY DR. G. A. MILLER.

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ALL the regular groups of these orders have been determined by Cole and Glover and by Hölder.* It is the object of this paper to determine all the transitive groups that are simply isomorphic to these regular ones. As every substitution group of a given order is simply isomorphic to one and only one regular group, we shall thus find all the possible non-regular transitive groups whose orders are the products of any three prime numbers. At the same time we shall be

* A regular substitution group may be said to be determined by the simply isomorphic abstract or operation group and *vice versa*.

able to observe a number of important properties of the given regular groups. We shall first develop the theorem upon which our method is based.

Let $a_1, a_2, a_3, \cdots, a_n$ and $1, s_2, s_3, \cdots, s_g$ represent the elements and the substitutions, respectively, of a transitive group G. The g substitutions may be arranged as follows:

$$\begin{array}{lllll} 1 & s_2 & s_3 \ \cdots\cdots\cdots & s_m & a_1 \\ t_2 & s_2 t_2 & s_3 t_2 \ \cdots\cdots\cdots & s_m t_2 & a_2 \\ t_3 & s_2 t_3 & s_3 t_3 \ \cdots\cdots\cdots & s_m t_3 & a_3 \\ \vdots & \vdots & \vdots \ \cdots\cdots\cdots & \vdots & \vdots \\ t_n & s_2 t_n & s_3 t_n \ \cdots\cdots\cdots & s_m t_n & a_n \end{array}$$

Where the line represented by a_α contains all the substitutions of G that replace a_1 by a_α. In particular, the substitutions of the first line do not contain a_1. It should be observed that the same letters are used to represent the elements of G, and also the lines in this arrangement of its substitutions. The reason for this will appear presently.

By multiplying all the substitutions of the line a_α into any one of the substitutions which replace a_α by a_β, we obtain each of the substitutions of the line a_β. Hence we obtain merely a new arrangement of the lines by multiplying all the substitutions of G into any one of its substitutions.* This new arrangement may be obtained from the given arrangement by transforming the a's representing the lines by the substitution which has been multiplied. By multiplying each of the substitutions of G by all its substitutions, we do not only obtain the identical substitution group in the a's representing the lines, but *we obtain identical substitution in every case*, that is, the rearrangement of the a's is always exactly represented by the substitution which has been multiplied.

It may be observed that we have not assumed that the line a_1 contains all the other elements of G. Our remarks, therefore, apply to all the cases when the first line is represented by any letter that does not occur in it, and the following lines by the letters which replace this in any one of their substitutions. By choosing one of the possible letters to represent one line, we determine all the others. When the group represented by a_1 is of degree $n-1$, it is more convenient to represent its n conjugates by the given letters and notice that each of the substitutions of G transforms these n conjugates in exactly the same way as it transforms its elements. As this notation is not as general as the pre-

* DYCK, *Mathematische Annalen*, vol. 22, p. 90.

ceding, we shall not employ it, and shall therefore not consider it at this place.

The given permutations of the lines in the given arrangement of the substitutions of G are independent of the elements by which G may be represented. We obtain the same permutations if we use for the s's the corresponding substitutions in any simple isomorphic group G', the simple isomorphism having been established in any one of the possible ways. The subgroup of G' which corresponds to the line a_1 can evidently not contain any operations that generate any self-conjugate subgroup of G' besides identity and the substitution which corresponds to t_a and is used to generate a new line in the arrangement of the substitutions G' must differ from any of the preceding substitutions in this arrangement. Conversely, if the substitutions of any group are so arranged that the first line contains a subgroup which includes no substitutions that generate any self-conjugate subgroup besides identity and if the following lines are formed in the given manner, these lines will be permuted according to a simply isomorphic substitution group when each of the substitutions is multiplied by every substitution of the group.* Hence we have the

THEOREM I. *The number of the transitive substitution groups that are simply isomorphic to a given group is equal to the number of its systems of subgroups that satisfy the two conditions:* (1) *Each system includes all the subgroups that are brought in correspondence when the group is made simply isomorphic to itself in every possible way, and* (2) *A subgroup that belongs to such a system does not include any operations that generate any self-conjugate subgroup with the exception of identity. All these transitive groups may be derived from the given group.*

In order that the applications which we shall make of this theorem may be more easily followed we shall employ it to determine the transitive groups that are simply isomorphic to (abc) all (de). There are only two systems of subgroups that satisfy the given conditions, viz.: 1 and the system which includes (ab). To obtain the group that corresponds to the former, *i. e.*, the simply isomorphic regular group we may write the substitutions as follows:

1	a_1	ab	a_4	de	a_7	$ab.de$	a_{10}
abc	a_2	ac	a_5	$abc.de$	a_8	$ac.de$	a_{11}
acb	a_3	bc	a_6	$abc.de$	a_9	$bc.de$	a_{12}

* DYCK, loc. cit.

By multiplying all of these substitutions into each of the two generators ab, ac, de we obtain the following two arrangements, a substitution (line) being represented by the corresponding letter:

$$\begin{array}{cccc} a_4 & a_1 & a_{10} & a_7 \\ a_6 & a_3 & a_{12} & a_9 \\ a_5 & a_2 & a_{11} & a_8 \end{array} \qquad \begin{array}{cccc} a_{11} & a_8 & a_5 & a_2 \\ a_{10} & a_7 & a_4 & a_1 \\ a_{12} & a_9 & a_6 & a_3 \end{array}$$

The substitutions that correspond to these arrangements are respectively:

$$a_1a_4 \cdot a_2a_6 \cdot a_3a_5 \cdot a_7a_{10} \cdot a_8a_{12} \cdot a_9a_{11}$$
$$a_1a_{11} \cdot a_2a_{10} \cdot a_3a_{12} \cdot a_4a_8 \cdot a_5a_7 \cdot a_6a_9$$

Hence these are generators of the regular group which is simply isomorphic to (abc) all (de). To obtain the transitive group which corresponds to the latter of the two systems we may write the substitutions of (abc) all (de) as follows:

$$\begin{array}{cccc} 1 & ab & a_1 & de & ab.de & a_4 \\ abc & ac & a_2 & abc.de & ac.de & a_5 \\ acb & bc & a_3 & acb.de & bc.de & a_6 \end{array}$$

By multiplying all these substitutions into the same two generators as before, we obtain the following two arrangements:

$$\begin{array}{cccc} a_1 & a_4 & a_5 & a_2 \\ a_3 & a_6 & a_4 & a_1 \\ a_2 & a_5 & a_6 & a_3 \end{array}$$

The substitutions of the lines that correspond to these arrangements are:

$$a_2a_3 \cdot a_5a_6 \qquad a_1a_5 \cdot a_2a_4 \cdot a_3a_6$$

Hence these are generators of the transitive group of degree six which is simply isomorphic to (abc) all (de).

Having called attention to the fundamental ideas employed in the following investigation we proceed to state in what sense we shall use several terms which are not always used in exactly the same sense. The term subgroup shall not include either identity or the entire group. Two subgroups of the same group are said to be transform subgroups when one can be transformed into the other by any operation that transforms the group into itself. With respect to regular groups or operation groups this is equivalent to saying that two subgroups are transform when they correspond in any one of the possible simple isomorphisms of

the groups to themselves.* Two or more self-conjugate subgroups may, therefore, be transform. The concepts, transform subgroups and characteristic subgroups, are clearly corresponding extensions of the concepts, conjugate subgroups and self-conjugate subgroups respectively.

It should be observed that the two definitions which Frobenius gives† for a characteristic subgroup (which are equivalent with respect to regular and operation groups), are not equivalent with respect to substitution groups in general. As may be inferred from what has been stated above, we adopt the former of these two definitions for the characteristic subgroup in substitution groups. It may happen that the characteristic and transform subgroups of a given group are identical with its self-conjugate and conjugate subgroups. This is, for instance, the case in the only non-regular transitive group, whose order is the product of two prime factors, as is evident from the fact that this group contains only one self-conjugate subgroup and that all its other subgroups constitute a single system of conjugate subgroups.

Our problem is clearly not as general as that embraced in the given theorem, since we have only to find the transitive groups which are simply isomorphic to given regular groups. It may, therefore, be convenient to replace the theorem by the corollary which we shall directly employ, using the terms as they have been defined above.

COROLLARY. *The number of transitive groups that are simply isomorphic to a regular group (G) is equal to the number of such systems of transform subgroups of G as do not include any self-conjugate subgroup of G. All these groups may be directly obtained from G.*

The non-regular transitive groups whose order is p^3.

Since the degree of these groups is p^3 they are included in Sylow's determination‡ of all the transitive groups of degree p^3 and order p^α. For the sake of completeness and simplicity we shall re-determine them by means of the given corollary. In the special case when $p = 2$, it is well known that $(abcd)_8$ is the only group of this kind. In other words, there is only one regular group of order 8 that contains subgroups which do not include substitutions that generate a

* FROBENIUS, *Sitzungsberichte der Berliner Akademie*, 1895, p. 184.

† Ibid, pp. 183 and 184.

‡ *Acta Mathematica*, vol. 11, p. 202. Cf. *Annals of Mathematics*, 1896, p. 156.

self-conjugate subgroup, and all these subgroups belong to the same system of transforms; in fact, they are even conjugate.

When $p > 3$ the two non-commutative regular groups contain substitutions of order p which are not commutative to all the substitutions of the entire group. Since a self-conjugate subgroup of order p in a group of order p^a contains no substitution that is not commutative to all the substitutions of the entire group, each of the two non-commutative regular groups must be simply isomorphic to at least one transitive group of degree p^2. It remains to prove that each of them is simply isomorphic to only one such group, or, in other words, that each of them contains only one system of transform subgroups that does not include any self-conjugate subgroup.

Each of these groups may be supposed to contain the non-cyclical group of order p^2 as head. The tail of one of them is composed of substitutions of order p^2, and that of the other is composed of substitutions of order p. Since each of them must be isomorphic to a commutative group with respect to its only self-conjugate subgroup of order p,* each of the substitutions of the other subgroups of order p must be transformed, by all the substitutions of the group, into itself multiplied by all the substitutions of the self-conjugate subgroup. Hence we see that any substitution in either of these groups transforms one of the substitutions of every non-self-conjugate subgroup of order p, to which it is not commutative, in exactly the same way as it transforms any given substitution of such a subgroup.

We can now see directly by means of the following elementary theorem that each of the given non-commutative regular groups of order p^3 contains only one system of transform subgroups, satisfying the given condition, and is, therefore, simply isomorphic to only one transitive group of degree p^2.

THEOREM II. *If two groups are placed in simple isomorphism we may add two new corresponding operations (s_1 and s_2), one to each, and thus obtain a simple isomorphism between two larger groups, provided:* (1) *s_1 and s_2 transform corresponding generating operations of the two simply isomorphic groups into corresponding operations, and* (2) *the first power of either s_1 or s_2 that occurs in one of the isomorphic groups corresponds to the same power of the other.*

Since each of the non-commutative groups of order p^3 is

* *Quarterly Journal of Mathematics*, vol. 28, p. 267.

generated by any two of its substitutions that are not commutative, we may obtain generating substitutions of each of the two possible non-regular transitive groups of order p^3 by multiplying all the substitutions of each regular non-commutative group of order p^2 into two of its substitutions that are non-commutative and observing how the lines, with respect to some non-self-conjugate subgroup of order p, are permuted by this operation. If we arrange the work so that the first line contains one of the multiplied substitutions we evidently obtain a substitution whose degree is less than p^2 as one of the two generators.

The non-regular transitive groups whose order is pq^2.

Every group of order pq^2 contains either a self-conjugate subgroup of order q^2 or it contains such a subgroup of order p. When the same group contains a self-conjugate subgroup of each of these orders it is commutative and can therefore not be represented as a non-regular transitive group. We shall first determine the non-regular transitive groups which contain a self-conjugate subgroup of order q^2.

According to Sylow's theorem each of these groups contains only one system of conjugate subgroups of order p. From this and the given corollary we have that *every regular group of order pq^2 that contains no self-conjugate subgroup of order p is simply isomorphic to one and only one transitive group of degree q^2.* As none of the required groups can be of a prime degree it remains only to find those which are of degree pq. In other words, it remains to determine the number of systems of transform groups of order q in the regular groups of order pq^2 that contain no self-conjugate subgroup of order p.

As the subgroups of order q must be contained in the self-conjugate subgroup of order q^2* there can be only one such subgroup when the given self-conjugate subgroup is cyclical. That is, there is no non-regular group of degree pq and order pq^2 that contains a self-conjugate subgroup of order q^2. This is also directly evident. When the given self-conjugate subgroup of order q^2 is non-cyclical it contains $q+1$ subgroups of order q. We need to consider only the two cases when the remaining substitutions of the groups permute either $q-1$ or all of these subgroups. By means of Theorem II we readily find that there is in each of these cases only one system of transform subgroups of order q that does not include any self-conjugate subgroup.

* FROBENIUS, *Crelle's Journal*, vol. 101, p. 284.

Hence each of these regular groups is simply isomorphic to one and only one non-regular transitive group of degree pq.

We have now determined the numbers of the non-regular transitive groups that are simply isomorphic to the different regular groups of order pq^2 that contain a self-conjugate subgroup of order q^2. By employing the lists of the regular groups we may state results as follows:

Of degree q^2, p being odd, there are $\frac{1}{2}(p+5)$ groups when $q-1$ is divisible by p and there is only one such group when $q+1$ is divisible by p. When $p=2$ there are 3 such groups.

Of degree pq, p being odd, there are $\frac{1}{2}(p+1)$ groups when $q-1$ is divisible by p and there is only one group when $q+1$ is divisible by p. When $p=2$ there is only one such group. When neither $q=1$ nor $q+1$ is divisible by p there are no such groups.

It remains to find the non-regular transitive groups that are simply isomorphic to the non-commutative regular groups of order pq^2 which contain a self-conjugate subgroup of order p. Since p has primitive roots, there must be one and only one such group of degree p, when $p-1$ is divisible by q^2. When this condition is not satisfied there is no group of this degree. It now remains only to find the possible groups of degree pq.

One of the three regular groups which satisfy the given conditions contains no substitution of order q^2. This contains only two systems of transform subgroups of order q. One of these systems consists of a characteristic subgroup. Hence this regular group is simply isomorphic to only one transitive group of degree pq. The other two of the given regular groups contain substitutions of order q^2.

The one which transforms the substitutions of the self-conjugate subgroup of order p according to the cyclical group of order $q-1$ contains only commutative substitutions of order q. The other, which transforms the substitutions of the self-conjugate subgroup of order p according to the cyclical group of order q^2 contains one conjugate system of groups of order q. Hence this is simply isomorphic to a transitive group of degree pq while the preceding is not simply isomorphic to such a group. The last is also simply isomorphic to one group of degree p as has been observed above.

Hence we have that there are two transitive groups of degree pq and order pq^2 that contain a self-conjugate subgroup of order p and there is one such group of degree p, when $p-1$ is divisible by q^2. When $p-1$ is divisible by q

but not by q^2 only one of these three groups occurs. This is of degree pq. When $p-1$ is not divisible by q there is no such group. This completes the determination of the non-regular transitive groups of degree pq^2. The generating substitutions can be found by the method which has been explained above.

The non-regular transitive groups whose order is pqr.

We may assume $p > q > r$. Since every group of order pqr contains only one subgroup of order p * there can be no transitive group of degree qr and order pqr. All of the given groups that do not contain an operation of order qr include also a self-conjugate subgroup of order q.† From Sylow's theorem it follows that these groups contain only one subgroup of each of the orders p and q. Hence we have that *each of the non-commutative groups of order pqr that do not contain an operation of order qr is simply isomorphic to one and only one transitive group of a lower degree and that the degree of each of these groups is pq.* It remains to consider those groups of order pqr which contain operations of order qr.

Since p has primitive roots, there is one and only one group of degree p and order pqr, provided $p-1$ is divisible by qr. When this condition is not satisfied, there is no such group. We see directly that this group contains more than one subgroup of each of the orders q and r. It is therefore simply isomorphic to a transitive group of degree pr, and also to one of degree pq.

If we establish a simple isomorphism between r non-cyclical transitive groups of order pq, and also between q non-cyclical transitive groups of order pr, we obtain two heads, to each of which we may add a substitution which merely interchanges the systems of intransitivity, and thus obtain the two remaining non-commutative groups of order pqr that contain substitutions of order qr. From this construction it is evident that the former contains a self-conjugate subgroup of order r, and is simply isomorphic to only one transitive group of degree pr while the latter contains a self-conjugate subgroup of order q and is simply isomorphic to only one transitive group of degree pq. As the non-regular transitive groups occur under the same conditions as the simply isomorphic regular groups, the determination of the non-regular transitive groups of order pqr is complete.

* COLE and GLOVER, *American Journal of Mathematics*, vol. 15, p. 215.
† HÖLDER, *Mathematische Annalen*, vol. 43, p. 370.

Summary.

We have now found all the possible non-regular transitive groups whose orders are the products of three prime factors by employing, in most cases, the regular groups of these orders. The results are as follows:

Order.	Degree.	Number.	Conditions.
p^3	p^2	2	when $p > 2$.
		1	when $p = 2$.
pq^2	p	1	when $p - 1$ is divisible by q^2.
	q^2	$\frac{p+5}{2}$	when $p > 2$ and $q - 1$ is divisible by p.
		1	when $p > 2$ and $q + 1$ is divisible by p.
		3	when $p = 2$.
	pq	$\frac{p+1}{2}$	when $p > 2$ and $q - 1$ is divisible by p.
		2	when $p - 1$ is divisible by q^2.
		1	when $p - 1$ is divisible by q but not by q^2.
		1	when $p > 2$ and $q + 1$ is divisible by p.
		1	when $p = 2$.
pqr	p	1	when $p - 1$ is divisible by qr.
	pr	2	when $p - 1$ is divisible by qr.
		1	when $p - 1$ is divisible by q but not by r.
	pq	$r + 2$	when $p - 1$ is divisible by qr and $q - 1$ is divisible by r.
		$r + 1$	when $p - 1$ is divisible by r but not by q and $q - 1$ is divisible by r.
		2	when $p - 1$ is divisible by qr but $q - 1$ is not divisible by r.
		1	when $p - 1$ is divisible by r but not by q and $q - 1$ is not divisible by r.
		1	when $p - 1$ is not divisible by r but $q - 1$ is divisible by r.

PARIS, *December* 7, 1896.

NOTE ON THE INTEGRATION OF A UNIFORMLY CONVERGENT SERIES THROUGH AN INFINITE INTERVAL.

BY PROFESSOR THOMAS S. FISKE.

[Read at the January meeting of the Society, 1897.]

IT is well known that if the terms of a uniformly convergent series

$$u_1(x) + u_2(x) + \cdots + u_n(x) + \cdots$$

are integrable through a *finite* interval $[a, b]$, the series defines a function $f(x)$ which is integrable through $[a, b]$, and its integral can be obtained by term by term integration, that is

$$\int_a^b f(x)\,dx = \int_a^b u_1(x)\,dx + \cdots + \int_a^b u_n(x)\,dx + \cdots$$

If, however, the interval considered is *infinite*, neither of these results is necessary.

Professor Osgood in his paper on uniform convergence and double limits, published in the BULLETIN for November, illustrates by an example the case where such a series is non-integrable through an infinite interval. It may be of interest to some of the readers of his paper to see the following example of a uniformly convergent series which is integrable through an infinite interval, but the integral of which cannot be obtained by summing the integrals of its terms.

The series in question is*

$$-2xe^{-x^2} = \sum_1^\infty -2x\left(\frac{1}{n^2}\,e^{-\frac{x^2}{n^2}} - \frac{1}{(n+1)^2}\,e^{-\frac{x^2}{(n+1)^2}}\right)$$

We have

$$s_n(x) = -2xe^{-x^2} + \frac{2x}{(n+1)^2}\,e^{-\frac{x^2}{(n+1)^2}},$$

$$r_n(x) = -\frac{2x}{(n+1)^2}\,e^{-\frac{x^2}{(n+1)^2}},$$

* This series was obtained by making certain modifications in a series used by DARBOUX in his "Mémoire sur les fonctions discontinues," *Annales de l'Ecole Normale*, 2 series, vol. 4 (1875), p. 84.

$$|r_n(x)| \leqq \frac{\sqrt{2}}{(n+1)\sqrt{e}}.$$

The uniformity of convergence follows from the fact that the second member of the last relation is independent of x.

By integrating this series term by term from 0 to x, we obtain

$$\sum_1^\infty \left(e^{-\frac{x^2}{n^2}} - e^{-\frac{x^2}{(n+1)^2}} \right),$$

which converges to

$$e^{-x^2} - 1 = \int_0^x -2xe^{-x^2}\,dx$$

only when x is finite, being equal to zero when x is infinite.

In this connection it may be observed that the integration of an infinite series through an infinite interval is a question of triple limits. Uniformity of convergence on the part of the given series enables us to write

$$\lim_{b=\infty} \int_a^b \lim_{n=\infty} s_n(x)\,dx = \lim_{b=\infty} \lim_{n=\infty} \int_a^b s_n(x)\,dx,$$

but it does not enable us to write

$$\lim_{b=\infty} \lim_{n=\infty} \int_a^b s_n(x)\,dx = \lim_{n=\infty} \lim_{b=\infty} \int_a^b s_n(x)\,dx.$$

A sufficient condition for the latter relation would be the uniform convergence in the infinite interval of the series resulting from the term by term integration of the given series. This condition, however, has the theoretical disadvantage of not being immediately applicable to the given series. Jordan has given* a sufficient condition which is free from this disadvantage, viz., that $r_n(x)$ should be expressible in the form $\varepsilon_n \varphi(x)$, $\varphi(x)$ being a positive function which has a finite integral in the infinite interval, and ε_n converging uniformly to zero when n approaches infinity.

COLUMBIA UNIVERSITY,
January 30, 1897.

* Cours d'Analyse, 2d edition, vol. 2 (1894), p. 62.

SHORTER NOTICES.

Ebene und räumliche Geometrie des Masses von DR. L. HUEBNER, Professor am Gymnasium zu Schweidnitz. Zweite Ausgabe. Leipzig, B. G. Teubner, 1895. 8vo. 16 and 340 pp.

This is an introduction to Circular and Hyperbolic Functions, with applications to Mensuration, of an unconventional kind. For instance, physical applications are sought out, such as pendulum motion and the paths of comets. The mensuration problems are extended to the finding of the volume of an ellipsoidal cap and analogous volumes, by aid of Cavalieri's principle. And the chapter on spherical Trigonometry is followed by one on spherical "Geometrie der Lage," which touches on the sphero-conic. The reason for speaking of spherical ellipses *and* hyperbolas does not appear.

Again, as to method, the book is in striking contrast to the stereotyped elementary trigonometry in refusing to attach the signs + and − to opposite lengths. It is supposed that the learner, familiar with these signs in their arithmetical use, will be confused by this new geometrical use. It would seem better to give all necessary preliminary explanation rather than to ban this excellent illustration of negative quantity, which seems to us to find its proper place in actual teaching at the outset of Algebra.

We note that the symbol θ is used for $1/0$ in contradistinction to $\infty = 1/\varepsilon$. Thus we have $\tan 90° = \theta$. This has an anarchistic look, but the point of view is not explained.

A theorem referred to as Feuerbach's, in connection with the triangle, is not the celebrated one, but a quite trivial thing which might have occurred to anyone.

And, to make a last, but important exception, the power-series for the sine and cosine, if at this stage they are proved at all, should be proved with some rigor.

At the same time, this independent book might well give suggestions to one wishing to reform his teaching of elementary Trigonometry; and the references show that several German teachers are in a state of dissatisfaction with the teaching of this subject by others. F. MORLEY.

The Scientific Papers of John Couch Adams. Vol. I. Edited by W. G. ADAMS, Sc. D., F. R. S. With a memoir by J. W. L. GLAISHER, Sc. D., F. R. S. Cambridge University Press, 1896. 4to. liv+502 pp.

Amongst the "collected works" of noted mathematicians

which are now being issued, those of J. C. Adams will hold an important and unusual position, inasmuch as the chief interest will attach to his unpublished work. The objection which he had against the publication of an investigation until he had put it into as complete a form as possible, is well known, and it is a matter for regret that much valuable work has, on this account, remained inaccessible for a long time. His published papers are comparatively few in number, the whole 62 of them appearing in the first volume; of these about half consist of the mere numerical results of long calculations, of addresses and astronomical observations. His brother, Professor Gryllis Adams, who edits the series, states that the manuscripts include numerous reseaches on Legendre's and Laplace's coefficients, terrestrial magnetism and astronomy. These, together with some of his lectures, will be issued in succeeding volumes.

A very readable biographical notice by Dr. Glaisher is appended, containing a straight-forward account of Adams' life and work. Dr. Glaisher has not fallen into the common error of giving a mass of irrelevant details, although the space devoted to the controversy concerning the discovery of Neptune is perhaps excessive, particularly as Adams took absolutely no part in it. The relative merits of Adams and Leverrier are now completely settled, and a full argument of Adams' claims is scarcely necessary. The account of the second great discussion—that on the secular acceleration of the Moon's mean motion—is also fully given. From the nature of the question, Adams was compelled to take some part in this, and the arguments on both sides are set forth in an interesting manner. It is curious that two of the fiercest scientific discussions of this century should have raged round the head of one of the most retiring and least pugilistic of men.

Of Adams as a lecturer, Dr. Glaisher has little to say, and yet his lectures, generally on some subject of mathematical astronomy, were often full of original matter. In a course on the lunar theory, which the writer attended some years ago, much of the matter given was entirely new, and previously published work was always cast into a new form. During the whole of the course, the blackboard was not used once. Long and complicated mathematical formulæ were entirely dictated and in such a way that his hearers had no difficulty in taking them down accurately. The results might almost be published as they stand, so finished are they in form.

The mathematical productions of the Pitt Press are too well known to need comment. It is sufficient to say here that the present volume is similar in form to Cayley's Collected Works. A portrait of Adams, as he appeared later in life, forms the frontispiece.

ERNEST W. BROWN.

NOTES.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York on Saturday afternoon, February 27, at half past three o'clock, the Vice-President, Professor R. S. WOODWARD, in the chair. There were sixteen members present. On recommendation of the Council the following persons, nominated at the preceding meeting, were elected to membership: Mr. ARTHUR BERRY, King's College, Cambridge, England; Professor JOHN EIESLAND, Thiel College, Greenville, Pa.; Mr. PAUL RENNO HEYL, Clinton Liberal Institute, Fort Plain, N. Y.; Professor ANNIE LOUISE MACKINNON, Wells College, Aurora, N. Y.; Mr. MILTON BROCKETT PORTER, Harvard University, Cambridge, Mass.; Professor DAVID ANDREW ROTHROCK, University of Indiana, Bloomington, Ind.; Professor WILLIAM EDWARD STORY, Clark University, Worcester, Mass.; Professor AUGUST LUDWIG PAUL WERNICKE, State College of Kentucky, Lexington, Ky.; Mr. JAMES KELSEY WHITTEMORE, Harvard University, Cambridge, Mass. Two nominations for membership were received.

A communication was received from the Council recommending certain amendments to the Constitution and By-Laws of the Society. Printed copies of the proposed changes will be mailed by the Secretary to members of the Society.

The following papers were read:

(1) Professor E. B. VAN VLECK: "Polynomial solutions of differential equations."

(2) Professor MAXIME BÔCHER: "On certain methods of Sturm and their application to the roots of Bessel's functions."

(3) Dr. VIRGIL SNYDER: "Lines common to four linear complexes."

In the absence of Professor Bôcher and Dr. Snyder, their papers were read by Professor Thomas S. Fiske.

A REVISED list of the officers and members of the AMERICAN MATHEMATICAL SOCIETY, corrected to January 1, 1897,

has been issued by the Secretary. The same pamphlet contains also the Constitution and By-Laws of the Society and the reports of the Treasurer and Librarian for 1896.

At the January meeting of the Council of the American Mathematical Society, a committee consisting of President Newcomb, Professors Fiske, Baker and White, and the Secretary, was appointed to make arrangements for the next Summer Meeting of the Society. This meeting will be held at Toronto, Canada, on Monday and Tuesday, August 16 and 17. It will thus intervene between the adjournment to Toronto of the Detroit Meeting of the American Association for the Advancement of Science and the opening of the Toronto Meeting of the British Association for the Advancement of Science. The committee will be very glad to receive suggestions from members of the Society in regard to the arrangements which are to be made.

An International Congress of Mathematicians is announced to be held at Zürich on August 9th, 10th and 11th of the present year. The committee on arrangements comprises representatives from Austria, England, France, Germany, Italy, Russia, Sweden, Switzerland and the United States. Dr. G. W. Hill is the American member. Correspondence regarding the Congress should be addressed to Professor C. F. Geiser, Küsnacht-Zürich, Switzerland.

The following is an abstract of the preliminary circular issued by the committee: "The question of an International Congress of Mathematicians has been agitated for several years. In view of the success attained by international coöperation in other fields, the desirability of similar action by mathematicians is universally conceded. After considerable preliminary correspondence in reference to the general plan, the question of the place of holding the Congress was decided in favor of Switzerland, as a country peculiarly adapted by situation, relations, and tradition for promoting international interests. The mathematicians of Zürich, although well aware of the responsibilities of the undertaking, have not hesitated to accept them, and will do their utmost for the success of the Congress. The meetings of the Congress will be held in the halls of the Züriçh Polytechnicum. The committee will hereafter issue a more elaborate programme and request the participation of mathematicians in the proceedings. It is appropriate, however, to observe in advance that both the scientific and the business transactions will naturally be confined to such questions as are of general interest and essential importance."

AMONG the prizes awarded by the French Academy of Sciences, at its annual meeting held December 21, 1896, were the following: *Grand Prix des Sciences Mathématiques* (10,000 francs), for perfecting in some important point the algebraic theory of substitutions of n letters, to EDMOND MAILLET. *Prix Bordin* (3,000 francs), for a contribution to the theory of geodesic lines, to JACQUES HADAMARD. *Prix Francoeur* (1,000 francs), awarded annually for discoveries or works furthering the progress of mathematical science, to A. VALSON, for his works in general, and in particular for editing the first eleven volumes of the collected works of Cauchy. *Prix Poncelet* (2,000 francs), awarded annually for the work published during the preceding ten years, which, in the judgment of the Academy, has been most useful to the progress of mathematical science, to PAUL PAINLEVÉ, for all of his mathematical works. *Prix Jean Reynaud* (10,000 francs), awarded annually in succession by each of the five academies for the work which in the field of that academy is the most important, from which prize, by a special provision of the founder, members of the Academy are not excluded, to HENRI POINCARÉ.

The subjects set by the Academy for the next mathematical prizes are as follows: *Grand Prix des Sciences Mathématiques* (10,000 francs), for extending the rôle played by divergent series in analysis, papers to be submitted by October 1, 1898. *Prix Bordin* (3,000 francs), for a memoir on the determination, properties and applications of systems of orthogonal curvilinear coördinates of n variables, the author to indicate as precisely as possible the degree of generality of the systems, papers to be submitted by October 1, 1898.

The *Prix Damoiseau* (1,500 francs), which was to have been awarded at this meeting for work in mathematical astronomy, has been postponed until June 1, 1897. A detailed statement of the subject was given in the BULLETIN for March, 1896.

As mathematicians are not very often honored with public monuments, a somewhat fuller account of the unveiling of the Lobachevsky monument at Kazàn, Russia, may perhaps be welcome. As stated before (BULLETIN, 2d series, Vol. III, p. 93), the unveiling of the statue took place on the 13th of September, 1896. The following account is derived from a special pamphlet, describing the celebration and adorned with a heliotype showing the monument (Kazàn, University Press, 1896), for which we are indebted to Professor A. Vasiliev. The monument, which is the work of

the St. Petersburg artist, Miss M. L. Dillon, consists of a bronze bust, about one meter in height, raised on a column of polished black granite, of 1½ meters; the pedestal is formed of two steps of unpolished gray granite; the total height is about 3 meters. The open square on which the statue is erected has received the name of Lobachevsky. The principal addresses made at the celebration were by Professors Suvòrov and Vasìliev, of the University of Kazàn, and by C. V. Shcherbakòv, President of the Physical and Astronomical Society of Nizhny-Nòvgorod (Lobachevsky's birthplace). In the evening the Physico-mathematical Society, of the University of Kazàn, held a solemn open session, at which a final account was rendered of the collection of the Lobachevsky fund. It was also stated that nine works had been received from France, Germany, Italy and America, competing for the Lobachevsky prize, which will be awarded for the first time on the 3d of October, 1897; the third volume of Professor Sophus Lie's "Theorie der Transformationsgruppen" is among these works. At the same session reports were read on the following papers, contributed by foreign mathematicians for the celebration: HERMITE, On some expansions in infinite series, occurring in the theory of elliptic functions; G. B. HALSTED, Darwinism and non-Euclidean geometry; P. GIRARDVILLE, Theory of the flight of birds; C. A. LAISANT, On the curvature of plane curves; E. LEMOINE, (a) Two new resolutions of integers, (b) Geometrography or theory of the simplification of geometrical constructions, (c) Continuous transformation in the triangle and tetrahedron; J. NEUBERG, On Jacobi's problem; M. d'OCAGNE, On the representation of equations of the second degree with three variables by means of straight lines and circles. All these papers will be printed, in the original languages, in Nos. 3–4 of the sixth volume of the *Journal of the Physico-Mathematical Society* of the University of Kazàn.

PROFESSOR O. SCHLÖMILCH, the founder of the *Zeitschrift für Mathematik und Physik*, has withdrawn from the editorship of this journal, which he has conducted for 41 years. Dr. R. MEHMKE, professor of mathematics in the Polytechnic School at Stuttgart, takes his place, while Professor M. CANTOR, of the University of Heidelberg, will continue to have charge of the "literarisch-historische Abteilung."

PROFESSOR E. PICARD has in press a work on the theory of functions of two imaginary variables.

L. Voss, of Hamburg, Germany, announces the publication of HELMHOLTZ's Lectures on the electro-magnetic theory of light, edited by A. KÖNIG and C. RUNGE; this will constitute the 5th of the proposed six volumes of Helmholtz's Lectures on theoretical physics.

DR. ARNOLD EMCH, assistant in graphics in the University of Kansas, has been called to a professorship of mathematics in the Polytechnic School at Biel, Switzerland.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BOHANNAN (R. D.). Simple proof of a fundamental theorem in the theory of functions. (*American Journal of Mathematics*, Vol. 19, 1 p.).

BURALI-FORTI (C.). Le classi finite. (*Accademia reale delle scienze di Torino*, 1896–97.) Torino, Clausen, 1896. 8vo. 21 pp.

BURALI-FORTI (C.). Sopra un teorema del Sig. G. Cantor. (*Accademia reale delle scienze di Torino*, 1896–97.) Torino, Clausen, 1896. 8vo. 11 pp.

CAYLEY (A.). The collected mathematical papers. (In 13 volumes.) Vol. XI. Cambridge (New York, Macmillan), 1896. 4to. 16 and 643 pp. $6.25

DUHAMEL. Tableau donnant les carrés et les racines carrées des nombres entiers. Paris, 1896. 8vo.

EMCH (A.). Projective groups of perspective collineations in the plane, treated synthetically. [Diss.] Lawrence, Kansas, Journal Press, 1896. 8vo. 35 pp.

GUNDELFINGER (S.). Tafeln zur Berechnung der reellen Wurzeln sämtlicher trinomischer Gleichungen. Hinzugefügt sind vierstellige Additions-, Subtractions- und Briggische Logarithmen, sowie eine Interpolationstafel für alle Differenzen unter Hundert. Leipzig, Teubner, 1897. 4to. 4 and 15 pp. Mk. 1.40

LAURENT (H.). See PETERSEN (J.).

MÉRAY (C.). Leçons nouvelles sur l'analyse infinitésimale et ses applications géométriques. (En 4 volumes.) Partie III: Questions analytiques classiques. Paris, Gauthier-Villars, 1897. 8vo. Fr. 6.00

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